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Linear SISO Systems with Extremely Sensitive Zero Structure

Jordan Berg and Harry G. Kwatny

***Abstract*—If a system with regular system pencil and relative degree greater than one is perturbed, the relative degree will typically decrease, and new finite zeros will appear. These new zeros are singularly perturbed. This paper applies a new canonical parameterization to systems with singular system pencils. Such systems have undefined relative degree. In singular systems, new zeros also appear under small perturbation, but they are not necessarily singularly perturbed. Rather, these zeros may appear at any frequency.**

I. INTRODUCTION

It is well known that relative degree is a structurally unstable property. Consider the system

$$\mathbf{G}(s) = \frac{\varepsilon s + 1}{s^2}. \tag{1}$$

If $\varepsilon = 0$, the transfer function has relative degree two, but if $\varepsilon \neq 0$ it has relative degree one. The zero that appears as a result of the perturbation ε is at $-1/\varepsilon$. For ε arbitrarily small, the magnitude of the zero can be made arbitrarily large. That is, the zero structure is singularly perturbed. If ε is negative, the zero is in the right half-plane.

The question of what structures may arise to replace an unstable property under small perturbation is the subject of singularity theory. This paper applies tools from singularity theory to study the zero structure transitions, under small perturbation, of linear, time-invariant, single-input–single-output (SISO) control systems. Consider such a system

$$\dot{x} = Ax + bu \tag{2a}$$

$$y = cx + du \tag{2b}$$

where $A \in C^{n\times n}$, $b \in C^{n\times 1}$, $c \in C^{1\times n}$, $x \in C^n$, and $u, y, d \in R$. Denote this system by

$$\begin{bmatrix} A & b \\ c & d \end{bmatrix}. \tag{2c}$$

Given (2), define a matrix pencil called the *system matrix*

$$\Gamma(s) = \begin{bmatrix} sl - A & b \\ -c & d \end{bmatrix}. \tag{3}$$

A pencil is *singular* if it is nonsquare or has identically zero determinant and is *regular* otherwise [1]. Regular SISO systems are very familiar. Singular SISO systems are not. The transfer function of a singular SISO system is identically zero, so it is unlikely that any such control system would be singular by design. However, singular systems can and do arise as subsystems of parameter-dependent families of multi-input–multi-output (MIMO) systems [2], [3]. This paper first considers regular systems and recovers well-known results

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J. Berg is with the Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN 55455 USA.

H. G. Kwatny is with the Department of Mechanical Engineering and Mechanics, Drexel University, Philadelphia, PA 19104 USA.

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for that case. It then treats—for the first time—singular systems, using the same method.

Two pencils $\mathbf{M}(s)$ and $\mathbf{N}(s)$ are *strictly equivalent*, written $\mathbf{M}(s) \sim \mathbf{N}(s)$, if there exist invertible, constant matrices P and Q^{-1} such that $P\mathbf{M}(s)Q^{-1} = \mathbf{N}(s)$. The invariants of the system matrix of (2) under strict equivalence are the *invariant zero structure* [4]. For the purposes of this paper, the invariant zero structure will be considered synonymous with the zero structure. The zero structure is displayed by the classical Kronecker form [1]. In the context of control systems, however, a better choice is the equivalent Thorp–Morse form [5], [6]. Unlike the Kronecker form, the Thorp–Morse form retains the system structure of (3), so if the pencil (3) is in Thorp–Morse form, it is meaningful to also refer to the system (2c) as being in Thorp–Morse form.

Relative degree is an important aspect of the zero structure, and it is closely related to one set of Kronecker invariants—the infinite divisors. So it is not surprising, considering example (1), that the Kronecker form (and so the Thorp-Morse form) is structurally unstable in the sense that can be made rigorous [7], [8]. This structural instability presents a challenge to numerical analysts trying to calculate the Kronecker form of a pencil. Van Dooren [9] solves that problem essentially by deciding whether the pencil is sufficiently close to another pencil with unstable structure. If it is, it is given the Kronecker form of that pencil. Here, rather than moving a pencil from a *less* degenerate structure to a *more* degenerate structure, the question is what less degenerate structures may be found in the neighborhood of a highly degenerate structure. This paper and other works by the authors apply this point of view purely for analysis. However, other researchers are taking a similar approach to numerical computation. For details of this work see [12]–[14].

The Jordan form for square matrices is, like the Kronecker form, structurally unstable under similarity transformation. Arnold [15] has presented a structurally stable canonical form based on the Jordan form of a square matrix. Structural stability is achieved by inserting free parameters into the Jordan form. These parameters can locally represent any perturbation of the original matrix. Of course this in itself is not too remarkable, since simply adding a free parameter to every element would suffice, but Arnold derives the minimal such parameterization. Furthermore, each parameter appears only once. Motivated by this work, Berg and Kwatny have derived a similar canonical parameterization of the Kronecker and Thorp–Morse forms [10], [11], [16]. The property of this parameterized canonical form that makes it useful here is that it is a *versal unfolding*; that is, every invariant zero structure in a neighborhood of the nominal system can be reached for some value of the parameter vector [15]. Therefore, to study all possible behaviors of the invariant zeros under small perturbation, it is only necessary to study the canonical parameterization. Furthermore, the parameterization contains the fewest possible parameters, so the computational effort is significantly reduced.

Section II of this paper derives the familiar properties of nonsingular SISO systems, using a novel method. Section III applies this method to singular SISO systems and presents an interesting example.

II. SISO SYSTEMS WITH WELL-DEFINED RELATIVE DEGREE

In this section, the canonical parameterization is used to prove the well-known result that the new zeros of a generically perturbed linear SISO system must be singularly perturbed. For example, see [17] for a singular perturbation analysis of affine nonlinear systems with well-defined relative degree that includes the systems considered below.

For linear SISO systems, well-defined relative degree is equivalent to a well-defined transfer function. Consider a nonsingular SISO system with $m > n$ distinct zeros

$$\mathbf{G}(s) = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}. \tag{4}$$

The system pencil of any minimal realization of (4), in the Thorp–Morse canonical form [5], is

$$\left[\begin{array}{cccccccc}
 & & & & & & & 0 \\
 & & & & & & & 0 \\
 & H_{(n-m)} & & & & 0 & & \vdots \\
 & & & & & & & 1 \\
 & & & & z_1 & & \mathbf{0} & \\
 & 0 & & & & \ddots & & 0 \\
 & & & & \mathbf{0} & & z_m & \\
1 & \cdots & 0 & 0 & & 0 & & 0
\end{array}\right] \tag{5}$$

where $z_1, \cdots, z_m$ are the roots of the numerator polynomial, corresponding to the finite zeros of (4), H has ones on the first superdiagonal and zeros elsewhere, and the order of H corresponds to the relative degree of (4). The canonical unfolding of (5) has $n+1$ parameters

$$\left[\begin{array}{cccccccc}
 & & & & & & & 0 \\
 & & & & & & & 0 \\
 & H_{(n-m)} & & & & 0 & & \vdots \\
 & & & & & & & 1 \\
 & & & & z_1 + \lambda_{n-m} & & \mathbf{0} & \\
 & 0 & & & & \ddots & & 0 \\
 & & & & \mathbf{0} & & z_m + \lambda_n & \\
1 & \lambda_1 & \cdots & \lambda_{n-m-1} & & 0 & & \lambda_{n+1}
\end{array}\right]. \tag{6}$$

If $\lambda_{n+1} \neq 0$, then the relative degree is zero. Otherwise

$$\begin{aligned}
cb &= \lambda_{n-m-1} \\
cAb &= \lambda_{n-m-2} \\
&\vdots \\
cA^{n-m-2}b &= \lambda_1 \\
cA^{n-m-1}b &= 1.
\end{aligned}$$

So the relative degree is determined by the first nonzero parameter between λ_{n-m-1} and λ_1. Generically, with only a causality constraint, none of the parameters is zero, so the generic relative degree is zero. With the additional constraint $\lambda_{n+1} \equiv 0$, the generic relative degree is one. Thus, for any system of relative degree greater than one, typical perturbations will cause one or more zeros to appear in the transfer function.

What are the zeros resulting from the perturbation? Take (6) and consider the easier case, $\lambda_{n+1} \neq 0$. The perturbed system is strictly equivalent to

$$\left[\begin{array}{cccccccc}
0 & 1 & & & & & & \\
 & 0 & \ddots & & & & & \\
 & & & & & 0 & & 0 \\
\mathbf{0} & & \ddots & 1 & & & & \\
-\frac{1}{\lambda_{n+1}} & -\frac{\lambda_1}{\lambda_{n+1}} & \cdots & -\frac{\lambda_{n-m-1}}{\lambda_{n+1}} & & & & \\
 & & & & z_1' & & & \\
 & & 0 & & & \ddots & & 0 \\
 & & & & & & z_m' & \\
 & & 0 & & & 0 & & 1
\end{array}\right] \tag{7}$$

where $z_k' = z_k + \lambda_k$. The perturbed relative degree is zero with m of the finite zeros being the original finite zeros, perturbed by a small amount (that is, regularly perturbed). The new zeros are the $n-m$ roots of the equation

$$\lambda_{n+1}s^{n-m} + \lambda_{n-m-1}s^{n-m-1} + \cdots + \lambda_1 s + 1 = 0. \tag{8}$$

By Rouché's theorem [18], given a circle of radius R centered on the origin of the complex plane, (8) has the same number of zeros inside the circle as does the polynomial one (that is, none at all) if $|\lambda_{n+1}s^{n-m} + \lambda_{n-m-1}s^{n-m-1} + \cdots + \lambda_1 s| < 1$ for every s on the circle. So, for any R greater than one, by restricting each parameter to have magnitude less than $R^{-(n-m)/(n-m)}$, all the corresponding zeros must have magnitude greater than R. So the new zeros are singularly perturbed. Although the new zeros are large, they may be nonminimum phase, as shown by the simple example (1).

Now consider the case $\lambda_{n+1} \equiv 0$. Without loss of generality, let λ_{i+1} to λ_{n-m-1} be zero. The perturbed system is strictly equivalent to the following one:

$$\begin{bmatrix}
0 & 1 & & \mathbf{0} & & & & & & & & 0 \\
 & \ddots & \ddots & & & & & & & & & 0 \\
 & & \ddots & 1 & & & 0 & & & 0 & & \vdots \\
\mathbf{0} & & & 0 & & & & & & & & 1 \\
 & & & & 0 & 1 & & & & & & \\
 & 0 & & & & 0 & \ddots & & & 0 & & 0 \\
 & & & & \mathbf{0} & & \ddots & 1 & & & & \\
 & & & & -\frac{1}{\lambda_i} & -\frac{\lambda_1}{\lambda_i} & \cdots & -\frac{\lambda_{i-1}}{\lambda_i} & & & & \\
 & & & & & & & & z_1' & & & \\
 & 0 & & & & 0 & & & & \ddots & & 0 \\
 & & & & & & & & & & z_m' & \\
1 & 0 & \cdots & 0 & & & 0 & & & 0 & & 0
\end{bmatrix}. \tag{9}$$

The perturbed relative degree is $n - m - i$, with m of the finite zeros again being the regularly perturbed original finite zeros. The new zeros are the i roots of the equation

$$\lambda_i s^i + \lambda_{i-1}s^{i-1} + \cdots + \lambda_1 s + 1 = 0. \tag{10}$$

Once again, an appeal to Rouché's theorem reveals that the new zeros are singularly perturbed. Note finally that if finite zeros are repeated, the perturbation structure may be complicated [15], but the analysis above is not altered.

III. SISO SYSTEMS WITH UNDEFINED RELATIVE DEGREE

Not all SISO systems have well-defined relative degree. The "pathological" cases occur when the system matrix is singular [1]. Let a system be called *singular* if its system matrix is singular. Then a SISO system is singular if and only if its transfer function is identically zero. This situation may arise in a parameterized family of systems, particularly as a subsystem of a larger MIMO system [2], [3]. In this section, the canonical unfolding is applied to obtain a new result for singular systems.

The following system shows the general form of a singular SISO system [5]:

$$\begin{bmatrix}
A_{\mathrm{ff}} & 0 & 0 & 0 \\
0 & A_{\varepsilon\varepsilon} & 0 & b_\varepsilon \\
0 & 0 & A_{\eta\eta} & 0 \\
0 & 0 & c_\eta & 0
\end{bmatrix} \tag{11}$$

where A_{ff} is a square $m \times m$ matrix whose eigenstructure is the finite zero structure of the system, $A_{\varepsilon\varepsilon}$ is a square $\varepsilon \times \varepsilon$ matrix with ones on the superdiagonal and zeros elsewhere, $A_{\eta\eta}$ is a square $\eta \times \eta$ matrix with ones on the superdiagonal and zeros elsewhere, b_ε is a $\varepsilon \times 1$ vector with a one in the bottom element and zeros elsewhere, and c_η is a $1 \times \eta$ vector with a one in the first element and zeros elsewhere. The dimensions satisfy $m + \varepsilon + \eta = n$.

Consider the case $\varepsilon \geq 0$ and $\eta > 0$. The unfolding then has general structure as shown in (12) at the bottom of the page [11]. In particular, if ε is zero, then (12) becomes

$$\begin{bmatrix}
 & \tilde{A}_{\mathrm{ff}} & & & & & 0 & & 0 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n & 0 & 1 & & & \lambda_{n+1} \\
0 & 0 & \cdots & 0 & & \ddots & \ddots & & \lambda_{n+2} \\
\vdots & \vdots & \ddots & \vdots & & & \ddots & 1 & \vdots \\
0 & 0 & \cdots & 0 & & & & 0 & \lambda_{n+\eta} \\
 & 0 & & & 1 & 0 & \cdots & 0 & \lambda_{n+\eta+1}
\end{bmatrix}. \tag{13}$$

If $\eta = 0$ and $\varepsilon > 0$, the perturbed form is [11]

$$\begin{bmatrix}
 & & & & \lambda_1 & \\
\tilde{A}_{\mathrm{ff}} & 0 & & & \lambda_2 & 0 \\
 & & & & \vdots & \\
 & & & & \lambda_n & \\
 & 0 & 1 & & & 0 \\
0 & & \ddots & \ddots & & \vdots \\
 & & & \ddots & 1 & 0 \\
 & & & & 0 & 1 \\
0 & \lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{n+\varepsilon} & \lambda_{n+\varepsilon+1}
\end{bmatrix}. \tag{14}$$

If both ε and η are zero, the perturbed form is [11]

$$\begin{bmatrix}
z_1 + \lambda_1 & \mathbf{0} & & \lambda_{n+1} \\
 & \ddots & & \vdots \\
\mathbf{0} & & z_n + \lambda_n & \lambda_{2n} \\
\lambda_{2n+1} & \cdots & \lambda_{3n} & \lambda_{3n+1}
\end{bmatrix}. \tag{15}$$

$$\begin{array}{ccccccccccccc|l}
 & & & & & & & \lambda_1 & & & & & & \\
 & \tilde{A}_{\mathrm{ff}} & & & & 0 & & \lambda_2 & & 0 & & & 0 & \}\, n \\
 & & & & & & & \vdots & & & & & & \\
 & & & & & & & \lambda_n & & & & & & \\
 & & & & 0 & 1 & & & & & & & 0 & \\
 & 0 & & & & \ddots & \ddots & & & 0 & & & \vdots & \}\, \varepsilon \\
 & & & & & & \ddots & 1 & & & & & 0 & \\
 & & & & & & & 0 & & & & & 1 & \\
\lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{2n} & \lambda_{2n+1} & \lambda_{2n+2} & \cdots & \lambda_{2n+\varepsilon} & 0 & 1 & & & \lambda_{2n+\varepsilon+1} & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & \ddots & \ddots & & \lambda_{2n+\varepsilon+2} & \}\, \eta \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & & \ddots & 1 & \vdots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & & & 0 & \lambda_{2n+\varepsilon+\eta} & \\
 & 0 & & & & & 0 & & 1 & 0 & \cdots & 0 & \lambda_{2n+\varepsilon+\eta+1} & \}\, 1 \\
\hline
 & n & & & & & \varepsilon & & & \eta & & & 1 &
\end{array} \tag{12}$$

For each of these three cases, either $d \neq 0$ under typical perturbation, or if d is constrained to be zero, then $cb \neq 0$. So, as in the case of nonsingular systems, under typical perturbation singular systems are nonsingular and have relative degree zero or one.

Although the relative degree of the perturbed singular case behaves just as the nonsingular case does, the location of the resulting new zeros may be very different. Consider the following simple singular SISO system:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{16}$$

For this pencil, $\varepsilon = 1$, $\eta = 1$. The single input controls an unobservable subsystem, and the single output observes an uncontrollable subsystem. The transfer function is identically zero. The system has no isolated invariant zeros. Following (12), the canonical unfolding is

$$\begin{bmatrix} 0 & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 \\ 0 & 1 & \lambda_3 \end{bmatrix}. \tag{17}$$

Set $\lambda_3 \equiv 0$, corresponding to a strictly proper system, and consider the generic case $\lambda_2 \neq 0$. Then the Thorp–Morse form of the perturbed system is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{\lambda_1}{\lambda_2} & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{18}$$

By comparison to (5), $H = 0$, so the perturbed system has (as expected) relative degree one. Again comparing to (5), or referring to [4] or [5], it has a single finite zero at $-\lambda_1/\lambda_2$. Constrain λ_1/λ_2 to be real to ensure realizability. Then the perturbed system has a finite zero at $-\lambda_1/\lambda_2$. Now let $\lambda_1 = -\sigma\lambda_2$, where σ is any real number. Then for arbitrarily small perturbations, the perturbed system has a zero at σ, where σ is completely arbitrary. This is quite remarkable considering that the zeros created by perturbing the regular system had magnitude that approached infinity when the magnitude of the perturbation went to zero. Note that the analysis of affine systems in [17] explicitly omits systems without a well-defined relative degree such as (16).

The "flip side" of this example is also remarkable. That is, this singularity is structurally stable in two parameter families. Thus if a system "near" this singular system contains two independent variables, it is possible, even likely, that the resulting parameterized family will contain a system exhibiting this singularity.

IV. CONCLUSION

The zero structure of nonsingular SISO systems behaves in a well-understood way under perturbation. In such cases the relative degree is well defined. The relative degree is generically zero for proper systems and one for strictly proper systems. When a nongeneric system is slightly perturbed, one or more new zeros appear. These zeros are singularly perturbed. Singular SISO systems have identically zero transfer functions. They have undefined (or infinite) relative degree. These systems are structurally unstable and upon perturbation will become nonsingular. In this case, also, new zeros appear. These new zeros, however, are not necessarily singularly perturbed and may be at any frequency.

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